A property of diagrams of the trivial knot

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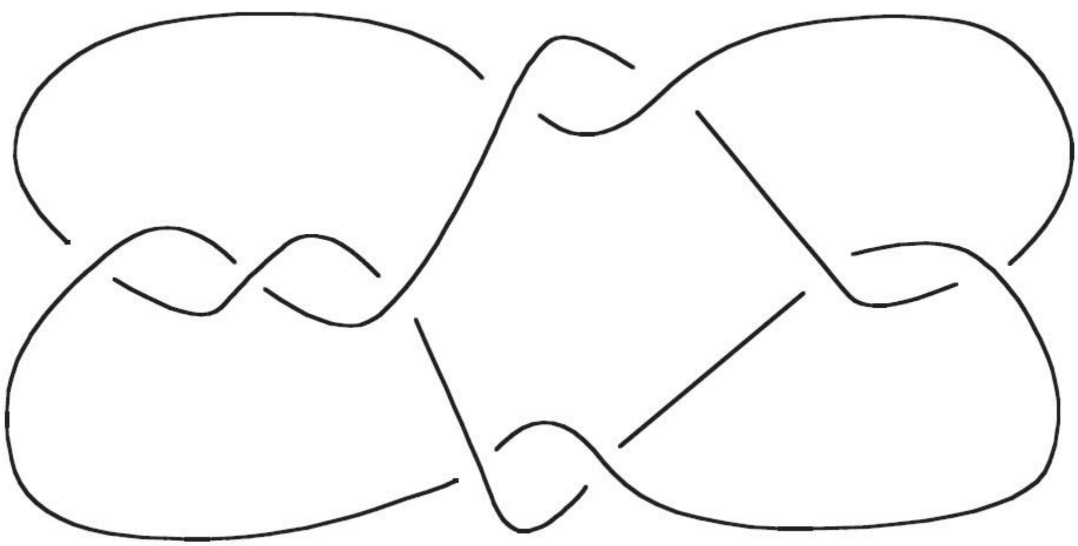
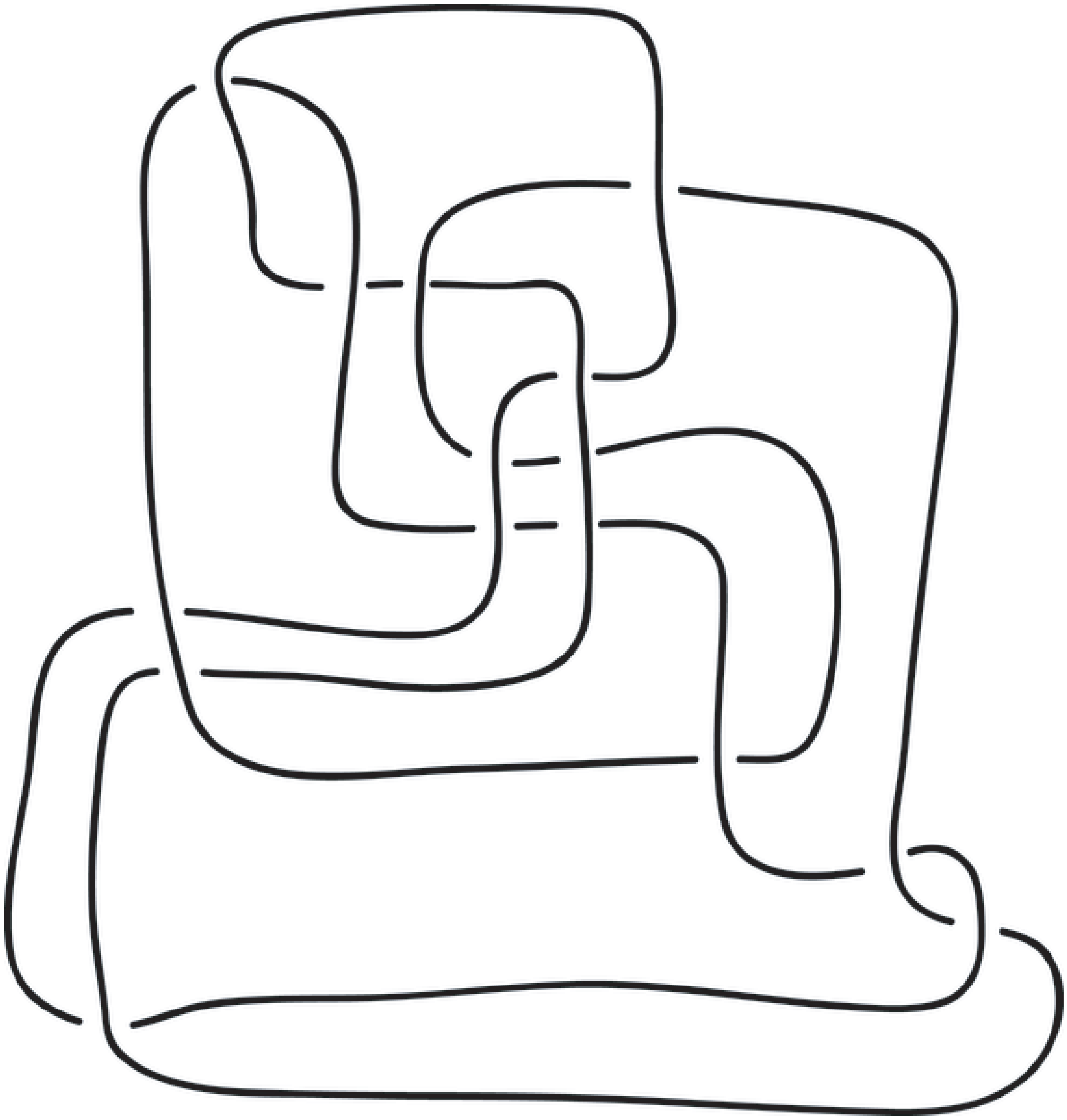


Figure 1: Goeritz's unknot.



Morwen Thistlethwaite's unknot

Figure 3.5. Wolfgang Haken's "Gordian knot."

Definition

K : trivial $\iff \exists S^2 \supset K$

$\iff \exists D^2$ s.t. $\partial D^2 = K$

Theorem (Papakyriakopoulos, 1957)

K : trivial $\iff \pi_1(S^3 - K) \cong \mathbb{Z}$

C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. **66** (1957) 1-26.

Theorem (Haken, 1961)

$\exists$ algorithm to decide whether K is trivial

W. Haken, *Theorie der Normalflachen*, Acta Math. **105** (1961) 245-375.

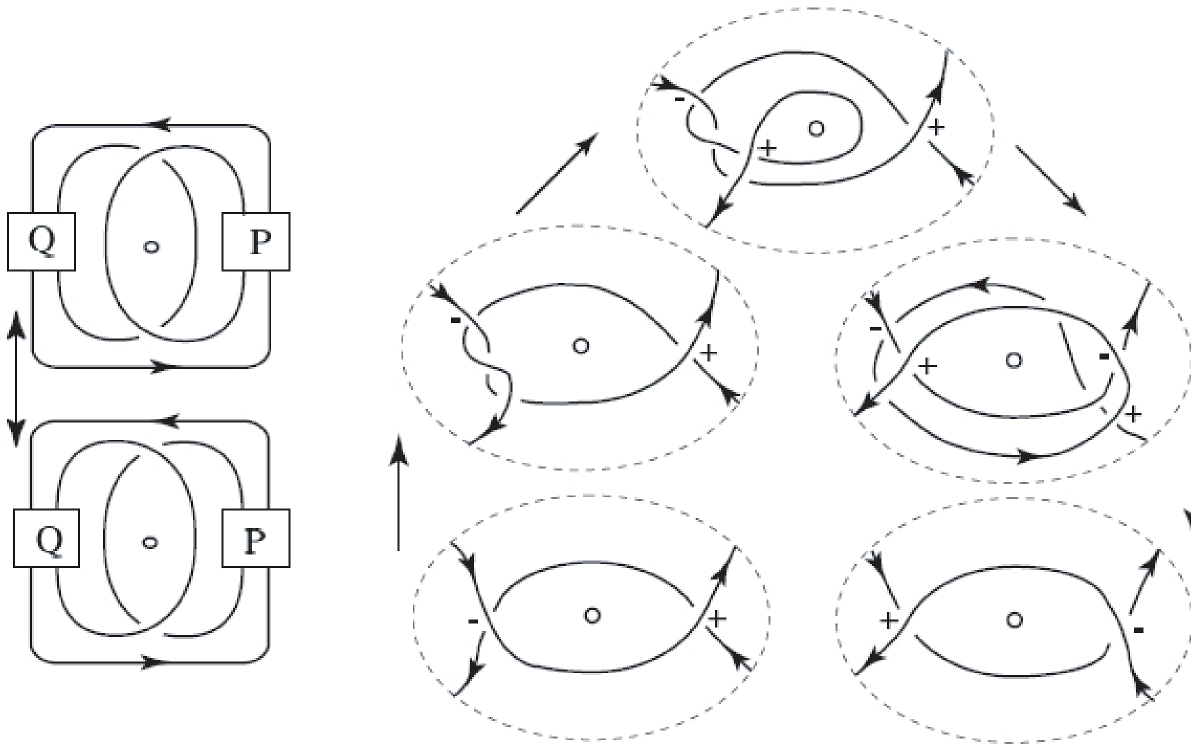
Braid presentation

Theorem (Birman-Menasco, 1992)

Every closed braid representative K of the unknot $\mathcal{U}$ may be reduced to the standard 1-braid representative U_1, by a finite sequence of braid isotopies, **destabilizations** and **exchange moves**.

Moreover there is a complexity function associated to closed braid representative in the sequence, such that each destabilization and exchange move is strictly complexity-reducing.

J. Birman and W. Menasco, *Studying Links Via Closed Braids V: The Unlink*, Trans. AMS **329** (1992) 585-606.



The left top and bottom sketches define the exchange move.

The right sequence of 5 sketches shows how it replaces a sequence of Markov moves which include braid isotopy, a single stabilization, additional braid isotopy and a single destabilization.

Thin position

Theorem (Scharlemann, 2004)

If the unknot is in **bridge position**, then either it is in thin position (and so has just a single minimum and maximum) or it may be made thinner via an isotopy that does not raise the width.

Question (Scharlemann)

Suppose $K \subset S^3$ is the unknot. Is there an isotopy of K to thin position (i.e. a single minimum and maximum) via an isotopy during which the width is never increasing?

M. Scharlemann, *Thin position in the theory of classical knots*, in the Handbook of Knot Theory, 2005.

Diagram

Theorem

Let D be a diagram without nugatory crossings of a knot K . If D is

- alternating, or
- positive, or
- homogeneous, or
- adequate,

then K is non-trivial.

R. H. Crowell, *Genus of alternating link types*, Ann. of Math. **69** (1959), 258-275.

K. Murasugi, *On the genus of the alternating knot II*, J. Math. Soc. Japan **10** (1958), 235-248.

J. v. Buskirk, *Positive links have positive Conway polynomial*, Springer Lecture Notes in Math. **1144** (1983), 146-159.

P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. **39** (1989) 535-552.

M. B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math. **93** (1998) 285-296.

Problem

What kind of property do diagrams of the trivial knot have?

Hereafter, we assume that a diagram is **I-reduced** and **II-reduced**, i.e. there is no part in a diagram whose crossing number can be reduced by a Reidemeister I-move and II-move.

Furthermore, we assume that a diagram is **prime**, i.e. which has at least one crossing and any loop intersecting it in two points cuts off an arc.

Tools

K : trivial $\iff$ $E(K)$: solid torus

Proposition

$\forall$ orientable surface ($\neq D^2$) properly embedded in the solid torus is

1. **compressible** or
2. incompressible and ∂-parallel annulus

Policy

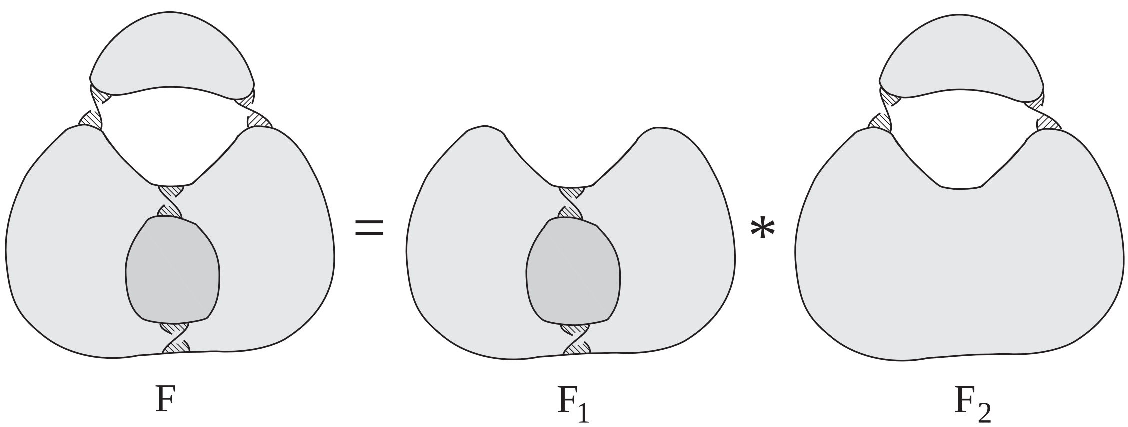
We do not let K bound a disk, but let K bound a surface except for a disk, and examine intersections of a **compressing disk** for the surface and **regions** of diagram.

Approach 1. canonical Seifert surface F

F is compressible.

Theorem (Gabai)

$F = F_1 * F_2$: Murasugi sum
F : compressible
$\Rightarrow F_1$: compressible or F_2 : compressible



Hence, we may assume that every Seifert circle is non-nested.

D. Gabai, *The Murasugi sum is a natural geometric operation*, Contemp. Math. **20** (1983) 131–143.

Approach 2. checkerboard surface F

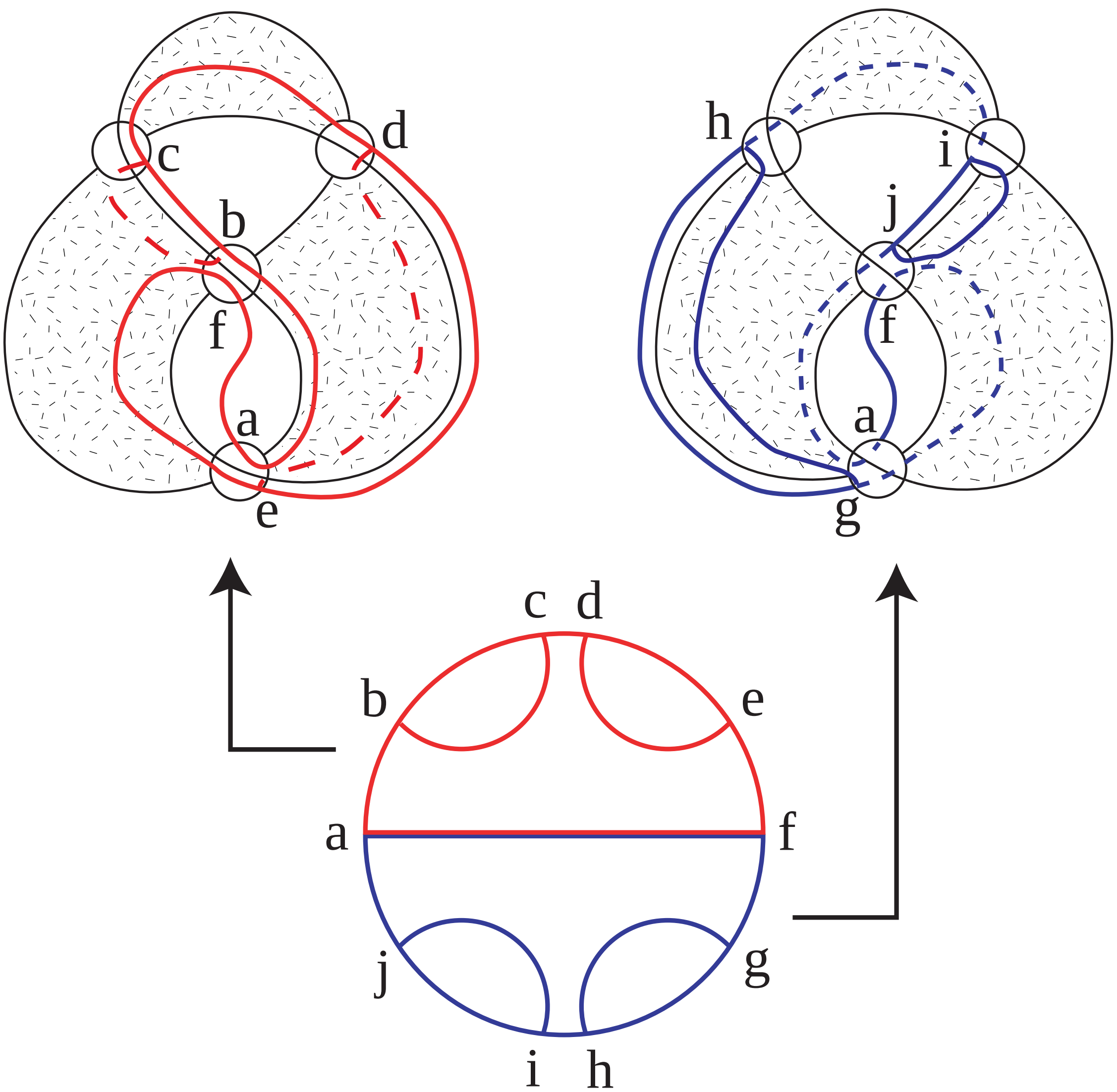
$\tilde{F} = F \tilde{\times} \partial I$
$\tilde{F}$ is compressible in the outside of $F \tilde{\times} I$.

D : compressing disk
$R = S^2 - F \tilde{\times} I$: complementary region of $F \tilde{\times} I$

We examine intersections of D and R.

We assume that the number of components of $D \cap R$ is minimal.

Example. 4-crossing diagram of the right-handed trefoil without nugatory crossings. The checkerboard surface is compressible.

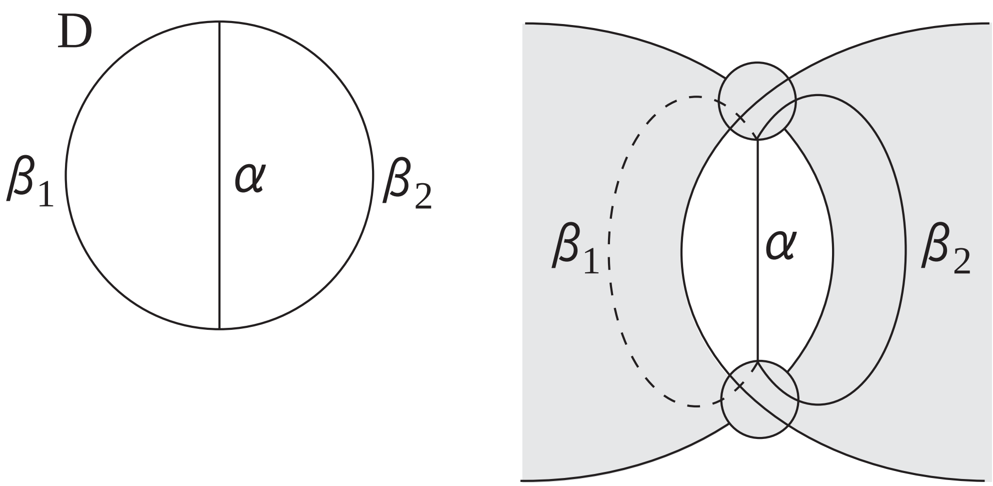


In this case, $|D \cap R| = 5$.

(Continuation of Proof)

If $|D \cap R| = 0$, then ∂D is not essential in $\tilde{F}$.

If $|D \cap R| = 1$, then

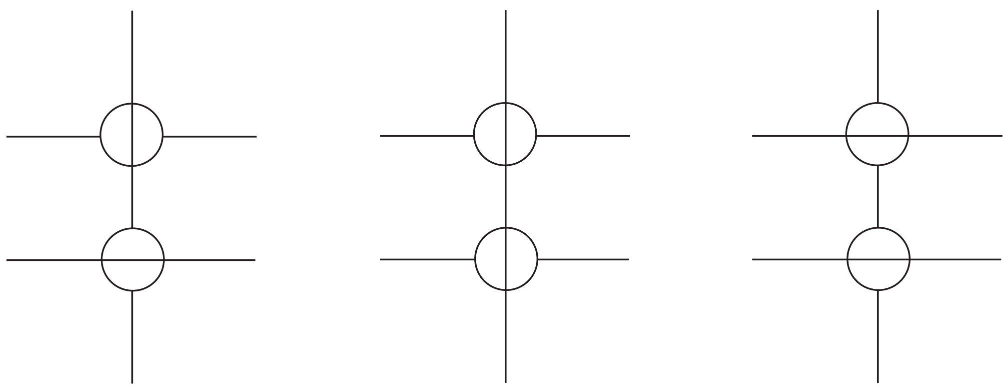


A diagram is II-reducible.

Hereafter, we assume that $|D \cap R| > 1$,
and focus on outermost arcs in D.

Claim

Any outermost arc α connects $\pm$-adjacent crossings.

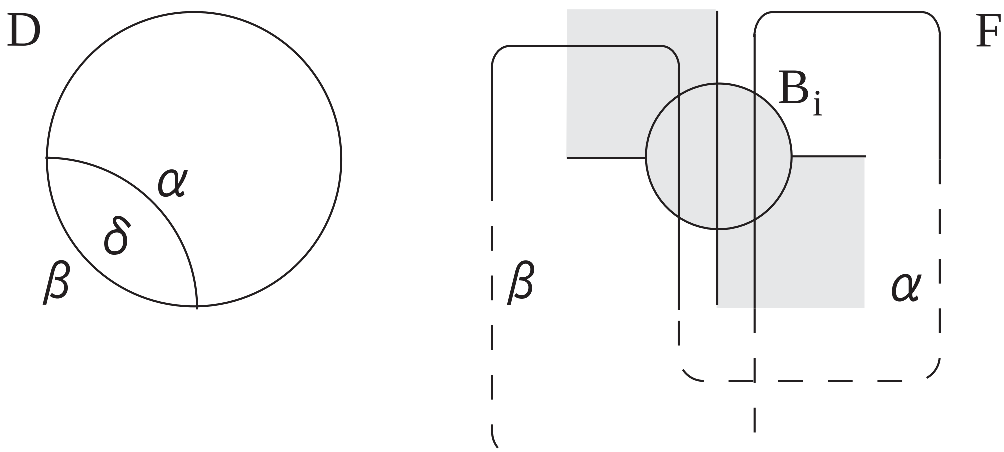


adjacent, $+$-adjacent, $-$-adjacent

Proof.

(Case1)

If α connects a same crossing, then

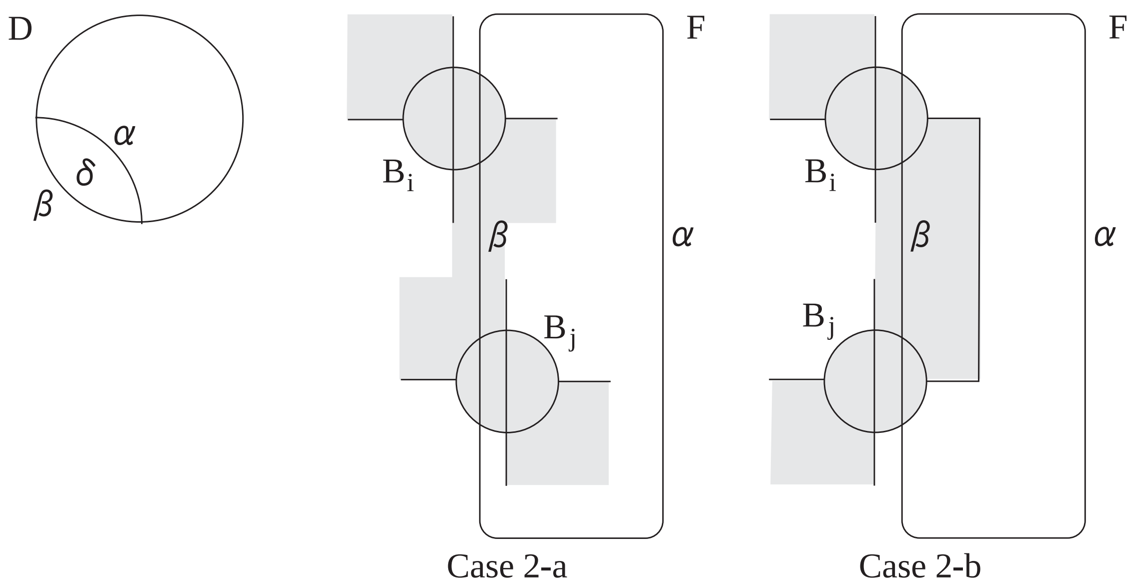


We have two loops obtained from arcs α and β, which intersects in one point.

This contradicts the Jordan Curve Theorem.

(Case 2)

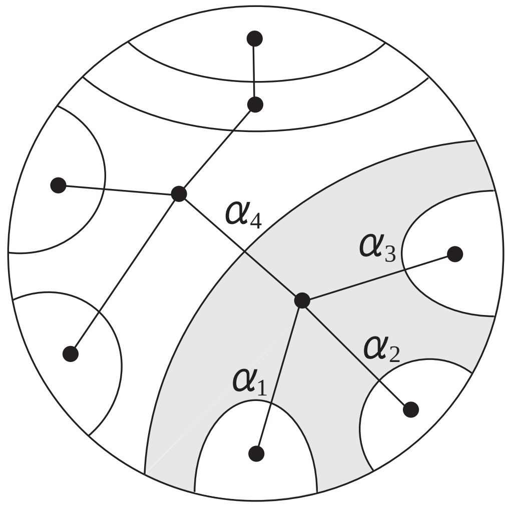
If α connects two different crossings, then



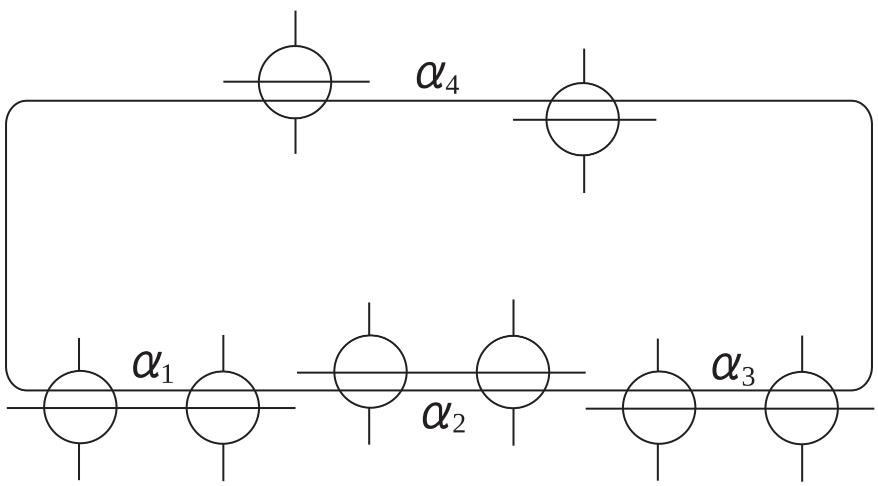
In Case 2-a, the diagram is composite.

In Case 2-b, the diagram is composite or two crossings are $\pm$-**adjacent**.

Next, we pay attention to an outermost fork.



Then by Claim, we have



We call such a loop as a boundary of a subdisk in D $\pm$-**Menasco loop**.

In summary,

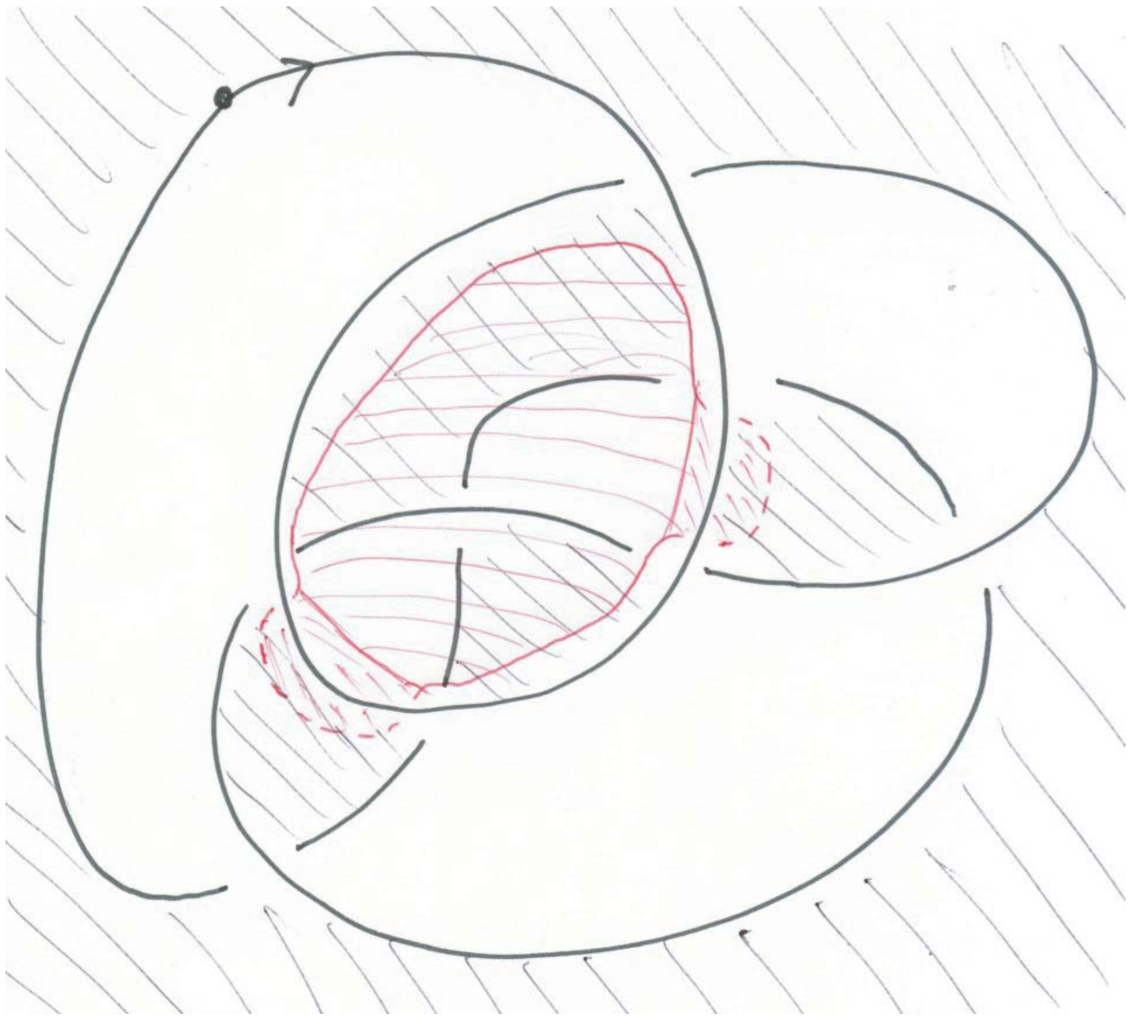
Theorem

Any I-reduced, II-reduced, prime diagram of the trivial knot has a $\pm$-Menasco loop passing through $2n$-crossings $c_1, c_2, \ldots, c_{2n}$, where $n \geq 2$ and c_{2i-1} is $\pm$-adjacent to c_{2i} for $i = 1, \ldots, n-1$.

Remark. In Theorem, we can take a compressing disk D so that ∂D does not pass through a one side of a crossing **more than once**.

Development. It is possible to state that for a checkerboard surface F, whether $\tilde{F}$ is compressible by means of **all $\pm$-Menasco loop** coming from **all subdisk** in D.

Application.



Any descending diagram gives the trivial knot. A loop appearing in a generalized Reidemeister move I forms a +-Menasco loop satisfying the condition in Theorem.

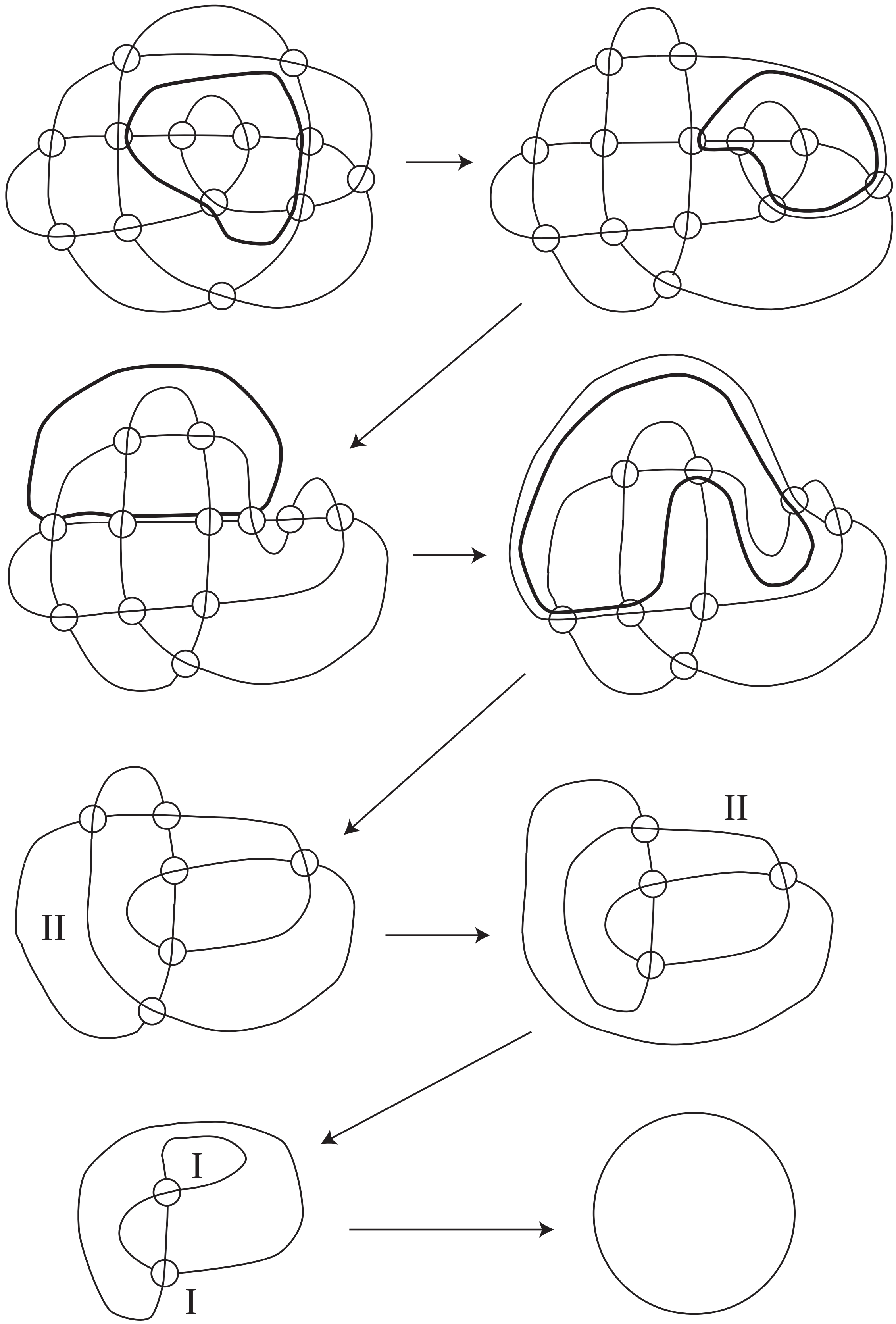
Next example is borrowed from Ochiai's book.

This diagram of the trivial knot has no r-wave for any $r \geq 0$.

At each stage, there exists a $\pm$-Menasco loop satisfying the condition in Theorem or it is not I-reduced or not II-reduced.

In the former case, a $\pm$-Menasco loop can be used to simplify the diagram if it has **successive three adjacent crossings**, and in the latter case, the crossing number can be reduced by a Reidemeister move I or II.

M. Ochiai, *Introduction to knot theory by computer*, Makino publisher, 1996. (In Japanese)


II
II
I
I

Final example is somewhat artificial.

This diagram is 2-almost alternating, that is, obtained from an alternating diagram by twice crossing changes on it.

There does not exist a $\pm$-Menasco loop satisfying the condition in Theorem. Hence, this knot is non-trivial.

Note that Tsukamoto characterized almost alternating diagarms of the trivial knot.

T. Tsukamoto, *The almost alternating diagrams of the trivial knot*, preprint available at http://lanl.arxiv.org/abs/math.GT/0605018.