ON UNIQUENESS OF ESSENTIAL TANGLE DECOMPOSITIONS OF KNOTS WITH FREE TANGLE DECOMPOSITIONS

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1. Introduction

Let B be a 3-ball and $t = t_1 \cup \ldots \cup t_n$ a union of mutually disjoint n arcs properly embedded in B. Then we call the pair (B, t) an *n-string tangle.* We say that an n-string tangle (B, t) is *trivial* if (B, t) is homeomorphic to $(D \times I, \{x_1, \ldots , x_n\} \times I)$ as pairs, where D is a disk and x_i is a point in $intD$ $(i = 1, \ldots , n)$. According to [1], we say that (B, t) is *essential* if $cl(\partial B - N(t))$ is incompressible and ∂-incompressible in $cl(B - N(t))$. And, according to [4], we say that (B, t) is *free* if $\pi_1(B - t)$ is a free group. We note that (B, t) is free if and only if $cl(B - N(t))$ is a handlebody ([3, 5.2]).

Let K be a knot in S^3 and S a 2-sphere in S^3 intersecting K in $2n$ points. Then the pair (S^3, K) is decomposed by S into two n-string tangles (B_1, t_1) and (B_2, t_2), and the union $(B_1, t_1) \cup_S (B_2, t_2)$ is called an *n-string tangle decomposition* of K. An n-string tangle decomposition $(B_1, t_1) \cup_S (B_2, t_2)$ is said to be *essential* (resp. *free*) if both (B_1, t_1) and (B_2, t_2) are essential (resp. free). We say that two n-string tangle decompositions $(B_1, t_1) \cup_S (B_2, t_2)$ and $(C_1, s_1) \cup_R (C_2, s_2)$ are *isotopic* if there exists an isotopy $f : S^2 \times I \to S^3$ of a 2-sphere S^2 in S^3 such that $f(S^2 \times 0) = S$, $f(S^2 \times 1) = R$ and $f((S^2 \cap K) \times I) \subset K$.

For a knot K, we define the *n-string tangle number* $T_n(K)$ of K as the number of essential n-string tangle decompositions of K modulo isotopy.

Then Gordon-Reid's result is stated as follows.

Theorem 1.1. [1] *Let K be a knot which admits an inessential free 2-string tangle decomposition. Then $T_n(K) = 0$ for any n.*

In this paper, we will show the following theorem that expands [7, Corollary 1.2].

Theorem 1.2. *Let K be a knot which admits an essential free 2-string tangle decomposition. Then $T_2(K) = 1$ and $T_n(K) = 0$ for all $n \neq 2$.*

2. Preliminaries

In this section, we consider how to prove Theorem 1.2.

According to [6], an n-string tangle (B, t) is said to be *indivisible* if for any disk D properly embedded in B intersecting t in one point in its interior, D cuts off a trivial

1-string tangle from (B, t). An n-string tangle decomposition $(B_1, t_1) \cup_S (B_2, t_2)$ of a knot in S^3 is said to be *indivisible* if both (B_1, t_1) and (B_2, t_2) are indivisible. Suppose that a knot in S^3 admits a divisible essential n-string tangle decomposition $(B_1, t_1) \cup_S (B_2, t_2)$, and that (B_1, t_1) is divisible by a disk D. Let (B_{11}, t_{11}) and (B_{12}, t_{12}) be the tangles divised by D from (B_1, t_1). Then we have the following proposition.

Proposition 2.1. *Both tangle decompositions* $(B_{11}, t_{11}) \cup (B_{12} \cup B_2, t_{12} \cup t_2)$ *and* $(B_{12}, t_{12}) \cup (B_{11} \cup B_2, t_{11} \cup t_2)$ *are essential tangle decompositions of* K.

This proposition says that any essential tangle decomposition of a knot can be divised into some indivisible essential tangle decompositions of the knot. Conversely, any essential tangle decomposition of a knot can be obtained from some indivisible essential tangle decompositions of the knot by 'tubing operations'. Therefore to prove Theorem 1.2, it is enough to prove the following theorem.

Theorem 2.2. *Let* K *be a knot in* S^3 *which admits an essential free* 2*-string tangle decomposition* $(B_1, t_1) \cup_S (B_2, t_2)$. *Then any indivisible essential tangle decomposition is isotopic to* $(B_1, t_1) \cup_S (B_2, t_2)$.

3. Natures of free tangles

In this section, we study on natures of free 2-string tangles.

First, we review a result of Gordon-Reid and Morimoto.

Lemma 3.1. [1],[5] *Let* M *be an orientable closed* 3*-manifold with a genus two Heegaard splitting* (V_1, V_2). *If* M *contains a* 2*-sphere* S *such that each component of* $S \cap V_1$ *is a non-separating disk in* V_1 *and* $S \cap V_2$ *is incompressible and not* ∂*-parallel in* V_2*, then* M *has a lens space or* $S^2 \times S^1$ *summand.*

The following Lemmas 3.2 and 3.3 follow Lemma 3.1.

Lemma 3.2. *Let* (B, t) *be a free* 2*-string tangle and* S *a* 2*-sphere in* $\mathrm{int}B$ *intersecting* t *transversely. If* $S - t$ *is incompressible in* $B - t$*, then one of the following conclusions holds.*

(1) S *bounds a trivial* 1*-string tangle.*

(2) S *is isotopic rel.*t *to* ∂B.

Proof. Glue a 3-ball B' to B along their boundaries. Put $V_1 = B' \cup N(t; B)$ and $V_2 = cl(B - N(t; B))$. Then (V_1, V_2) is a genus two Heegaard splitting of the 3-sphere $B \cup B'$, each component of $S \cap V_1$ is a non-separating disk in V_1, and $S \cap V_2$ is incompressible in V_2. In consequence of this observations and Lemma 3.1, $S \cap V_2$ is ∂-parallel in V_2. Therefore $S \cap V_2$ is an annulus or a 2-sphere with four holes. In the formar case, we obtain the conclusion (1), and in the latter case, we obtain the conclusion (2). □

Lemma 3.3. *Let $(B, t_1 \cup t_2)$ be a free 2-string tangle, and let P be a planar surface properly embedded in B such that each component of ∂P separates two points of ∂t_i in ∂B for each $i = 1, 2$. If $P - (t_1 \cup t_2)$ is incompressible in $B - (t_1 \cup t_2)$, then one of the following conclusions holds.*

(1) P is an annulus in $B-(t_1 \cup t_2)$ which is isotopic to an annulus in $\partial B-(t_1 \cup t_2)$.

(2) P is a disk intersecting t_i in one point for each $i = 1, 2$ which is isotopic rel.$(t_1 \cup t_2)$ to a disk in ∂B.

(3) P is an annulus intersecting only one component of $t_1 \cup t_2$ in two points which is obtained from two disks D_1 and D_2 of (2) by a tubing operation, where D_1 and D_2 are isotopic to ∂B into the different directions.

Proof. Let C be a 3-ball and $E_1 \cup \ldots \cup E_{|\partial P|}$ be a union of mutually disjoint parallel $|\partial P|$ disks properly embedded in C. Glue C to B so that $\partial C = \partial B$ and $\partial(E_1 \cup \ldots \cup E_{|\partial P|}) = \partial P$. Put $V_1 = C \cup N(t_1 \cup t_2; B)$, $V_2 = cl(B - N(t_1 \cup t_2; B))$ and $S = P \cup E_1 \cup \ldots \cup E_{|\partial P|}$. Then (V_1, V_2) is a genus two Heegaard splitting of the 3-sphere $B \cup C$, each component of $S \cap V_1$ is a non-separating disk in V_1, and $S \cap V_2$ is incompressible in V_2. In consequence of this observations and Lemma 3.1, $S \cap V_2$ is ∂-parallel in V_2. Therefore we have the following cases.

(1) P is an annulus and disjoint from $t_1 \cup t_2$.

(2) P is a disk and intersects t_1 and t_2 in one point respectively.

(3) P is an annulus and intersects only one component, say t_1, of $t_1 \cup t_2$ in two points.

In cases (1) and (2), we have the conclusions (1) and (2) respectively. In case (3), since $S \cap V_2$ is ∂-parallel in V_2, there is an embedding $f : (S \cap V_2) \times I \to V_2$ such that $f((S \cap V_2) \times \{0\}) = S \cap V_2$ and $\partial f((S \cap V_2) \times I) - S \cap V_2 \subset \partial V_2$. Let a be the core of an annulus $N(t_2; B) \cap cl(B - N(t_2; B))$ with $a \subset int f((S \cap V_2) \times \{1\})$. Put $A = f(f^{-1}(a) \times I)$ and $\hat{A} = A \cup d$, where d is a disk bounded by a in $N(t_2; B)$ and intersects t_2 in one point. Then by compressing P along $\hat{A}$, we have two disks D_1 and D_2 of (2). Moreover the embedding $f|_{cl(S \cap V_2 - N(a; S \cap V_2))}$ shows that D_1 and D_2 are isotopic to ∂B into the different directions. $\square$

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2.

Let K be a knot in S^3 which admits an essential free 2-string tangle decomposition $(B_1, t_1) \cup_S (B_2, t_2)$, and let $(C_1, s_1) \cup_R (C_2, s_2)$ be an indivisible essential tangle decomposition of K. We may assume that $S \cap R = (S - K) \cap (R - K)$ consists of loops, and assume that $|S \cap R|$ is minimal among all 2-string tangle decompositions isotopic to $(C_1, s_1) \cup_R (C_2, s_2)$.

Claim 4.1. *If $|S \cap R| = 0$, then $(C_1, s_1) \cup_R (C_2, s_2)$ is isotopic to $(B_1, t_1) \cup_S (B_2, t_2)$.*

Proof. This is due to Lemma 3.2. □

From now on, we suppose that $|S \cap R| \neq 0$.

Claim 4.2. *$S \cap R$ consists of mutually parallel loops in $S - K$ that split four points of $S \cap K$ into pairs of two points in S.*

Proof. By the incompressibility of $R - K$ in $S^3 - K$ and the minimality of $|S \cap R|$, each component of $S \cap R$ is an essential loop in $S - K$. Further, by the indivisibility of $(C_1, s_1) \cup_R (C_2, s_2)$, no loop of $S \cap R$ bounds a disk D in S which intersects K in one point. Thus we have the conclusion of Claim 4.2. □

Since K is a connected simple closed curve, by Claim 4.2, each component of $R \cap B_i$ satisfies the hypothesis of Lemma 3.3 for either $i = 1$ or 2. If there is a component of $R \cap B_i$ which is of (1) or (2) in Lemma 3.3, then this contradicts the minimality of $|S \cap R|$. Hence any component of $R \cap B_i$ is of (3) in Lemma 3.3. Then we can find a disk D in B_i such that $D \cap t_i = intD \cap t_i =$one point and $D \cap (R \cap B_i) = \partial D \cap (R \cap B_i) = \partial D$. By the disk D and the indivisibility of $(C_1, s_1) \cup_R (C_2, s_2)$, we can isotop a component of $R \cap B_i$ of (3) so that it becomes of (2). This contradicts the minimality of $|S \cap R|$. These complete the proof of Theorem 2.2. □

5. THIN POSITION OF KNOTS WITH FREE TANGLE DECOMPOSITIONS

In this section, we remark about thin position of knots which admit free 2-string tangle decompositions.

First, we review the definition of thin position of knots. Let $\pm\infty$ be the north and south poles of S^3. Then $S^3 - \{\pm\infty\}$ is naturally homeomorphic to $S^2 \times R^1$, and we have an associated height function $h : S^3 - \{\pm\infty\} \to R^1$. Let K be a knot in S^3 and let $f = \{f_s\}(s \in [0, 1])$ be an ambient isotopy of S^3 such that $f_1(K) \subset S^3 - \{\pm\infty\}$ and $h|_{f_1(K)}$ is a Morse function. Choose a regular value t_i between each pair of adjacent critical values of $f|_{f_1(K)}$. Define the *width* of K *with respect to* f to be the sum over i of the [number of intersections of $f_1(K)$ with $h^{-1}(t_i)$], and denote it by $\omega_f(K)$. Define the *width* of K, $\omega(K)$, to be the minimum width of K with respect to f over all f. We say that K is in *thin position* if it is in a position which realizes its width.

We say that S is a *thin 2-sphere* for K with respect to h if $S = h^{-1}(t)$ for some t which lies between adjacent critical values x and y of h, where x is a minimum of K lying above t and y is a maximum of K lying below t. Define the *height* of K *with respect to* f to be the [number of thin 2-spheres for $f_1(K)$]+1, and denote it by $ht_f(K)$. Define the *height* of K, $ht(K)$, to be the maximum height of K with respect to f over all f with $\omega_f(K) = \omega(K)$.

Then Thompson has shown the following theorem.

Theorem 5.1. [8, Corollary 4] *Let K be a knot which admits an inessential free 2-string tangle decomposition. Then $ht(K) = 1$.*

By Theorem 1.2 and [2, Proposition 3.7], we have;

Theorem 5.2. *Let K be a knot which admits an essential free 2-string tangle decomposition. Then $ht(K) \leq 2$.*

Remark 5.3. [2, Example 5.1] *There exists a knot K in Theorem 5.2 such that $ht(K) = 1$.*

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